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Valentin Zagrebnov

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Valentin A. Zagrebnov

(Département de Mathématiques, Université d'Aix-Marseille and Institut de Mathématiques de Marseille - UMR 7353, France)

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Valentin.Zagrebnov@univ-amu.fr

We justify the Bogoliubov c-number approximation for the case of interacting Bose-gas in a *homogeneous random* media. To this aim we take into account occurrence of generalised (*extended/fragmented*) Bose-Einstein condensation in an infinitesimal band of low *kinetic-energy* modes, to generalise the c-number substitution procedure for this band of low-momenta modes.

1 Introduction

One of the key developments in the theory of the Bose gas, especially the theory of the low density gases currently at the forefront of experiment, is Bogoliubov's 1947 analysis [2], [3] of the many-body Hamiltonian by means of a c-number substitution for the most releVant operators in the problem. These are the zero-momentum mode boson operators, namely $b_0 \to z$, $b_0^* \to z^*$. Later this idea triggered a more general *The Approximating Hamiltonian Method* [6]. Naturally, the appropriate value of z has to be determined by some sort of consistency or variational principle, which might be complicated, but the concern is whether this sort of substitution is legitimate, i.e., error free.

The rigorous justification for this *substitution*, as far as calculating the pressure of interacting (superstable) boson gas is concerned, was done for the first time in the paper by Ginibre [10]. Later it was revised and essentially improved by Lieb-Seiringer-Yngvason (LSY) [15], [16]. In textbooks it is often said, for instance, that it is tied to the imputed "fact"that the expectation value of the *zero-mode* particle number operator $N_0 = b_0^* b_0$ is of order $V =$ volume. This was the *second* Bogoliubov ansatz: the Bose-Einstein condensation (BEC) *justifies* the substitution [26].

As Ginibre pointed out, however, that BEC has nothing to do with it. The z substitution still gives the right answer for any value of the Gibbs average of the operator N_0. On the other hand, the zero-mode translation invariant condensation (the *first* Bogoliubov ansatz) plays a distinguished role in the Bogoliubov Weakly Imperfect Bose theory [26].

The problem of justification becomes delicate in a (*bona fide*) homogeneous random external potential: first of all because of the translation invariance breaking and secondly because of the problem with nature of the *generalised* BEC for this case even for the perfect Bose-gas [11], [12]. The aim of the present note is to elucidate this problem for interacting boson gas in a homogeneous random potential following the LSY method. The later allows to simplify and make more transparent the arguments of [13] versus the *generalised* condensation *à la* Van den Berg-Lewis-Pulé [24].

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2 The Bogoliubov c-Number Approximation

2.1 Imperfect Bose-Gas

Let interacting bosons of mass m be enclosed in a *cubic* box $\Lambda = L \times L \times L \subset \mathbb{R}^3$ of the volume $V \equiv |\Lambda| = L^3$, with (for simplicity) periodic boundary conditions on $\partial\Lambda$: $t_\Lambda := (-\hbar^2 \Delta / 2m)_{p.b.c.}$. Let $u(x)$ be *isotropic* two-body interaction with *non-negative* Fourier transformation:

$$v\,(q) = \int_{\mathbb{R}^3} d^3x\; u(x)e^{-iqx},\;\; u \in \mathcal{L}^1(\mathbb{R}^3)$$

The second-quantized Hamiltonian of the *imperfect* Bose-gas acting as operator *in* the boson Fock space $\mathfrak{F} := \mathfrak{F}_{boson}(\mathcal{H} = \mathcal{L}^2(\Lambda))$ is

$$H_\Lambda = \sum_{k\in\Lambda^*} \varepsilon_k b_k^* b_k + \frac{1}{2V} \sum_{k_1,k_2,q\in\Lambda^*} v\,(q)\, b_{k_1+q}^* b_{k_2-q}^* b_{k_2} b_{k_1}$$

where (*dual*) set $\Lambda^* = \left\{k \in \mathbb{R}^3 : ,\, k_\alpha = 2\pi n_\alpha/L \text{ et } n_\alpha \in \mathbb{Z}, \alpha = 1,2,3\right\}$ and $\{\varepsilon_k\}_{k\in\Lambda^*} = \mathrm{Spec}(t_\Lambda)$. Here $\{\varepsilon_k = \hbar^2k^2/2m \geq 0\}_{k\in\Lambda^*}$ is the one-particle excitations spectrum. The *perfect* Bose-gas Hamiltonian and *particle-number* operators are

$$T_\Lambda := \sum_{k\in\Lambda^*} \varepsilon_k\, b_k^* b_k \quad , \quad N_k := b_k^* b_k \;, \quad N_\Lambda := \sum_{k\in\Lambda^*} N_k \;.$$

Here $\{b_k^*, b_k\}_{k\in\Lambda^*}$ are boson creation and annihilation operators in the one-particle eigenstates (kinetic-energy modes) verifying the CCR $[b_k, b_q^*] = \delta_{k,q}$:

$$\psi_k\,(x) = \frac{1}{\sqrt{V}}\; e^{ikx}\;\; \chi_\Lambda(x) \in \mathcal{H},\;\; k \in \Lambda^*$$

$$b_k := b\,(\psi_k) = \int_\Lambda dx\; \overline{\psi_k}\,(x)\, b\,(x) \;\; , \;\; b_k^* = (b\,(\psi_k))^*$$

Here $b^\#\,(x)$ are boson-field operators in the Fock space over $\mathcal{H}$.

2.2 Grand-Canonical (β,μ)-Ensemble

Recall that the grand-canonical (β,μ)-state generated by H_Λ on algebra of observables $\mathfrak{A}(\mathfrak{F})$ [20], is define by

$$\langle A\rangle_{H_\Lambda} := \mathrm{Tr}_{\mathfrak{F}}(e^{-\beta(H_\Lambda-\mu N_\Lambda)}\; A)/\mathrm{Tr}_{\mathfrak{F}} e^{-\beta(H_\Lambda-\mu N_\Lambda)} \;,\; A \in \mathfrak{A}(\mathfrak{F}) \;.$$

The grand-canonical pressure: $p[H_\Lambda](\beta,\mu) := (\beta V)^{-1} \ln \mathrm{Tr}_{\mathfrak{F}} e^{-\beta(H_\Lambda-\mu N_\Lambda)}$ corresponds to the temperature β^{-1} and to the chemical potential μ.

Example 1. For the *perfect* Bose-gas T_Λ one must put $\mu < 0$, then the expectation value of the particle number in mode k is

$$\langle b_k^* b_k \rangle_{T_\Lambda} := \frac{1}{e^{\beta(\varepsilon_k - \mu)} - 1} \ , \quad \varepsilon_k \geq 0 \ .$$

The expectation value of the *total* density of bosons in Λ is

$$\rho_\Lambda(\beta, \mu) := \frac{1}{V} \langle b_0^* b_0 \rangle_{T_\Lambda} + \frac{1}{V} \sum_{k \in \Lambda^* \setminus \{0\}} \langle b_k^* b_k \rangle_{T_\Lambda} = \frac{1}{V} \frac{1}{e^{-\beta\mu} - 1} + \rho_\Lambda(\beta, \mu)$$

Then the *critical* density (if finite) is define by the limit:

$$\rho_c(\beta) := \lim_{\mu \uparrow 0} \lim_{\Lambda \uparrow \mathbb{R}^3} \rho_\Lambda(\beta, \mu) < \infty$$

.

2.3 Conventional Bose-Einstein Condensation

For a fixed density ρ, let $\mu_\Lambda(\beta, \rho)$ be solution of the equation

$$\rho = \rho_\Lambda(\beta, \mu) \quad \Rightarrow \quad \rho \equiv \rho_\Lambda(\beta, \mu_\Lambda(\rho)) \qquad (\textit{always exists!}).$$

• *low* density : $\lim_\Lambda \ \mu_\Lambda(\rho < \rho_c(\beta)) = \mu_\Lambda(\rho) < 0$
• *high* density: $\lim_\Lambda \ \mu_\Lambda(\rho \geq \rho_c(\beta)) = 0$, and

$$\rho_0(\beta) = \rho - \rho_c(\beta) = \lim_\Lambda \frac{1}{V} \ \left\{ e^{-\beta\, \mu_\Lambda(\rho \geq \rho_c(\beta))} - 1 \right\}^{-1} \Rightarrow$$
$$\mu_\Lambda(\rho \geq \rho_c(\beta)) = -\frac{1}{V} \ \frac{1}{\beta(\rho - \rho_c(\beta))} + o(1/V) \ .$$

• Since $\varepsilon_k = \hbar^2 \sum_{j=1}^d (2\pi n_j / V^{1/3})^2 / 2m$, the BEC is in $\mathbf{k}$=$\mathbf{0}$-mode:

$$\lim_\Lambda \frac{1}{V} \left\{ e^{\beta(\varepsilon_{k \neq 0} - \mu_\Lambda(\rho))} - 1 \right\}^{-1} = 0 \ ,$$

This type of condensation based on the concept of the one-level macroscopic occupation is known as the *conventional* zero-mode (or *type I*) BEC [9], [17].

2.4 Generalised Bose-Einstein Condensation

This type of condensation was predicted by Casimir [8] and elucidated by Van den Berg-Lewis-Pulé in [22], [23], [24].

Let $\Lambda = L_1 \times L_2 \times L_3 = V^{\alpha_1} \times V^{\alpha_2} \times V^{\alpha_3}$, $\alpha_1 \geq \alpha_2 \geq \alpha_3 > 0$, and $\alpha_1 + \alpha_2 + \alpha_3 = 1$.

• The *Casimir box* (1968): Let $\alpha_1 = 1/2$, i.e. $\alpha_{2,3} < 1/2$.

Since $\varepsilon_{k_1,0,0} = \hbar^2(2\pi n_1/V^{1/2})^2/2m \sim 1/V$, then again the asymptotics of solution:

$$\rho \equiv \rho_\Lambda(\beta, \mu_\Lambda(\rho)) \Rightarrow \mu_\Lambda(\rho \geq \rho_c(\beta)) = -A/V + o(1/V), \; A \geq 0$$

$$\lim_\Lambda \left\{ \frac{1}{V} \frac{1}{e^{-\beta\mu_\Lambda(\rho)} - 1} + \frac{1}{V} \sum_{k \in \{\Lambda^* : n_1 \neq 0, n_2 = n_3 = 0\}} \frac{1}{e^{\beta(\varepsilon_k - \mu_\Lambda(\rho))} - 1} \right\}$$

$$= \rho - \rho_c(\beta) > 0 \; , \; \lim_\Lambda \frac{1}{V} \left\{ e^{\beta(\varepsilon_{k\neq 0} - \mu_\Lambda(\rho))} - 1 \right\}^{-1} \neq 0, \varepsilon_{k \neq 0} = \varepsilon_{k_1,0,0}$$

$$\lim_\Lambda \frac{1}{V} \left\{ e^{\beta(\varepsilon_{k\neq 0} - \mu_\Lambda(\rho))} - 1 \right\}^{-1} = 0, \; \varepsilon_{0,k_{2,3}\neq 0} = \hbar^2(2\pi n_{2,3}/V^{\alpha_{2,3}})^2/2m$$

The generalised *type II* BEC [23]:

$$\rho - \rho_c(\beta) = \lim_{L\to\infty} \frac{1}{V} \sum_{n_1 \in \mathbb{Z}} \left\{ e^{\beta(\hbar^2(2\pi n_1/V^{1/2})^2/2m - \mu_\Lambda(\rho))} - 1 \right\}^{-1}$$

$$= \sum_{n_1 \in \mathbb{Z}} \frac{\beta^{-1}}{\hbar^2(2\pi n_1)^2/2m + A} \Rightarrow \lim_\Lambda \frac{1}{V} \langle b_0^* b_0 \rangle_{T_\Lambda(\mu_\Lambda(\rho))} < \rho - \rho_c(\beta).$$

Here $A \geq 0$ is a *unique root* of the above equation. Note that BEC in the zero mode is less than the total amount of the condensation density.

For $\alpha_1 = 1/2$ the BEC is still mode by mode macroscopic, but it is infinitely *fragmented*. This type of BEC is also known as *quasi-condensate* and it was observed in the *rotating* condensate (2000) and in the condensate with *chaotic* phases (2008), [18].

• The *Van den Berg box* (1982): $\alpha_1 > 1/2$.

Proposition 1. There is no macroscopic occupation of *any* kinetic-energy mode:

$$\lim_\Lambda \frac{1}{V} \left\{ e^{\beta(\varepsilon_k - \mu_\Lambda(\rho))} - 1 \right\}^{-1} = 0.$$

This is the generalised BEC of *type III* [Van den Berg-Lewis-Pulé (1978)]. It occurs one-direction anisotropy $\alpha_1 > 1/2$ i.e. $\alpha_2 + \alpha_3 < 1/2$. Since $\varepsilon_{k_1,0,0} = (2\pi n_1/V^{\alpha_1})^2/2 \sim 1/V^{2\alpha_1}$, $2\alpha_1 > 1$, then the solution $\mu_\Lambda(\rho)$ has a *new* asymptotics:

$$\mu_\Lambda(\rho \geq \rho_c(\beta)) = -B/V^\delta + o(1/V^\delta),\ B \geq 0,\ \delta = 2\,(1-\alpha_1) < 1\ ,$$

$$0 < \rho - \rho_c(\beta) = (2\pi\beta)^{-1/2} \int_0^\infty d\xi\ e^{-\beta B \xi}\ \xi^{-1/2}\ .$$

Here parameter $B = B\,(\beta, \rho) > 0$ is a *unique* root of the equation:

$$\rho - \rho_c(\beta) = \frac{1}{\sqrt{2\beta^2 B(\beta,\rho)}}\ .$$

The generalised BEC of *type III* yields for the one-mode particle occupation

$$\lim_\Lambda \frac{1}{V} \langle N_k \rangle_{T_\Lambda} (\beta, \mu_\Lambda\,(\rho > \rho_c(\beta))) = 0 \text{ for all } k \in \{\Lambda^*\}\ .$$

For the *"renormalised"* k_1-modes occupation "density"one obtains:

$$\lim_\Lambda \frac{1}{V^{1-\epsilon}} \langle N_k \rangle_{T_\Lambda} (\beta, \mu_\Lambda\,(\rho > \rho_c(\beta))) = 2\beta\,(\rho - \rho_c(\beta))^2\,,$$

where $k \in \{\Lambda^* : (n_1, 0, 0)\}$ and $1 - \epsilon = \delta < 1$.

Definition 1.[24] In kinetic-energy modes the amount of the generalised BEC is defined as

$$\rho - \rho_c(\beta) := \lim_{\eta \to +0} \lim_\Lambda \frac{1}{V} \sum_{\{k \in \Lambda^*, \|k\| \leq \eta\}} \left\{ e^{\beta(\varepsilon_k - \mu_\Lambda(\beta,\rho))} - 1 \right\}^{-1}\ .$$

Remark 1. [22],[24] Saturation and ρ_m*-problem*: is it possible that there is a new critical density ρ_m such that $\rho_c \leq \rho_m \leq \infty$ and the *type III (or II)* condensation transforms into conventional *type I* BEC when $\rho \geq \rho_m$? The answer is positive. Recently the second critical density ρ_m was discovered for a cigar-type harmonic anisotropy [1]. There it was also proved that the *type I* and the *type III* condensations may coexist.

2.5 The Bogoliubov Theory and the Zero-Mode c-Number Substitution

The first Bogoliubov ansatz. If one expects that the Bose-Einstein condensation, which occurs in the mode $k = 0$ for the perfect Bose-gas, ***persists*** for a ***weak*** two-body interaction $u(x)$, then one can to truncate Hamiltonian: $H_\Lambda \to H_\Lambda^B$, and to keep in H_Λ^B only the ***most important condensate*** terms, in which at least ***two*** zero-mode operators b_0^*, b_0 are involved. This approximation gives the Bogoliubov Weakly Imperfect Bose-Gas (WIBG) Hamiltonian H_Λ^B [26].
The second Bogoliubov ansatz. Since for a large volume (thermodynamic limit) the ***condensate*** operators $b_0^*/\sqrt{V}$, $b_0/\sqrt{V}$ ***almost*** commute: $[b_0/\sqrt{V}, b_0^*/\sqrt{V}] = 1/V$, one may use ***substitutions***:

$$b_0/\sqrt{V} \to c \cdot \mathbb{I}\ ,\ b_0^*/\sqrt{V} \to c^* \cdot \mathbb{I}\ , \quad c \in \mathbb{C}\ ,$$

in the truncated grand-canonical WIBG Hamiltonian $H_\Lambda^B(\mu) := H_\Lambda^B - \mu N_\Lambda \to H_\Lambda^B(c,\mu)$ to produce a diagonalizable bilinear operator form.

2.6 The Zero-Mode c-Number Approximation

For the periodic boundary conditions on $\partial\Lambda$, let $\mathfrak{F}_0 := \mathfrak{F}_{boson}(\mathcal{H}_0)$ be the boson Fock space constructed on the one-dimensional Hilbert space $\mathcal{H}_0$ spanned by $\psi_{k=0}(x) = \chi_\Lambda(x)/\sqrt{V}$.
Let $\mathfrak{F}_0' := \mathfrak{F}_{boson}\left(\mathcal{H}_0^\perp\right)$ be the Fock space constructed on the orthogonal complement $\mathcal{H}_0^\perp$. Then $\mathfrak{F}_{boson}(\mathcal{H}) = \mathfrak{F}_{boson}(\mathcal{H}_0 \oplus \mathcal{H}_0^\perp)$ is isomorphic to the *tensor product*:

$$\mathfrak{F}_{boson}\left(\mathcal{H}_0 \oplus \mathcal{H}_0^\perp\right) \approx \mathfrak{F}_{boson}(\mathcal{H}_0) \otimes \mathfrak{F}_{boson}(\mathcal{H}_0^\perp) = \mathfrak{F}_0 \otimes \mathfrak{F}_0',$$

For any complex number $c \in \mathbb{C}$ the coherent vector in $\mathfrak{F}_0$ is

$$\psi_{0\Lambda}(c) := e^{-V|c|^2/2} \sum_{k=0}^{\infty} \frac{1}{k!} \left(\sqrt{V}c\right)^k (b_0^*)^k\, \Omega_0 = e^{(-V|c|^2/2 + \sqrt{V}c\, b_0^*)}\ \Omega_0\ ,$$

where Ω_0 is the vacuum of $\mathfrak{F}$. Notice that

$$\frac{b_0}{\sqrt{V}}\ \psi_{0\Lambda}(c) = c\ \psi_{0\Lambda}(c) \equiv c \cdot \mathbb{I}\ \psi_{0\Lambda}(c)\,.$$

Definition 2. The *c-number* Bogoliubov approximation of the grand-canonical Hamiltonian ($N_\Lambda := \sum_{k\in\Lambda^*} b_k^* b_k := b_0^* b_0 + N'_\Lambda$)

$$H_\Lambda(\mu) := H_\Lambda - \mu N_\Lambda \ , \ \mathrm{dom}(H_\Lambda(\mu)) \subset \mathfrak{F} \approx \mathfrak{F}_{boson}(\mathcal{H}_0) \otimes \mathfrak{F}_{boson}(\mathcal{H}_0^\perp)$$

is a *self-adjoint operator* $H_\Lambda(c,\mu)$ defined in $\mathfrak{F}'_0 = \mathfrak{F}_{boson}\left(\mathcal{H}_0^\perp\right)$, for any fixed vector $\psi_{0\Lambda}(c)$, by the closable sesquilinear form:

$$(\psi'_1, H_\Lambda(c,\mu)\psi'_2)_{\mathfrak{F}'_0} \equiv (\psi_{0\Lambda}(c) \otimes \psi'_1, H_\Lambda(\mu)\,\psi_{0\Lambda}(c) \otimes \psi'_2)_{\mathfrak{F}} \ ,$$

for vectors $(\psi_{0\Lambda}(c) \otimes \psi'_{1,2}) \in$ *form-domain* of the operator $H_\Lambda(\mu)$.

Remark 2. Since $(b_0/\sqrt{V})\ \ \psi_{0\Lambda}(c) = c\cdot\mathbb{I}\ \ \psi_{0\Lambda}(c)$, the *c-number* approximation is *equivalent* to *substitutions*:

$$b_0/\sqrt{V} \to c\cdot\mathbb{I}\ ,\ b_0^*/\sqrt{V} \to c^*\cdot\mathbb{I}$$

in the Hamiltonian
$H_\Lambda(\mu) \to H_\Lambda(c,\mu) =: H'_\Lambda(z) - \mu(|z|^2\,\mathbb{I} + N'_\Lambda)\ ,\quad z := c\,\sqrt{V}.$

2.7 Exactness of the c-Number Approximation

Definition 3. The grand-canonical pressure for Hamiltonian $H_\Lambda(\mu)$ and for its *c-number* Bogoliubov approximation $H'_\Lambda(z,\mu)$, are defined by:

$$p_\Lambda(\mu) := \frac{1}{\beta V}\ln \mathrm{Tr}_{\mathfrak{F}} \exp[-\beta H_\Lambda(\mu)]$$
$$p'_\Lambda(\mu) := \frac{1}{\beta V}\ln \int_{\mathbb{C}} d^2 z \mathrm{Tr}_{\mathfrak{F}'_0} \exp[-\beta H'_\Lambda(z,\mu)]$$

Proposition 2.(Variational Principle) [10], [15].

$$e^{\beta V p_\Lambda(\mu)} \geq \int_{\mathbb{C}} d^2 z \mathrm{Tr}_{\mathfrak{F}'_0} \exp[-\beta H'_\Lambda(z,\mu)] \geq$$
$$\sup_\zeta \mathrm{Tr}_{\mathfrak{F}'_0} \exp[-\beta H'_\Lambda(\zeta,\mu)] =: e^{\beta V p_{\Lambda,max}(\mu)}$$

Proposition 3.

$$\lim_\Lambda p_\Lambda(\mu) = \lim_\Lambda p'_\Lambda(\mu) = \lim_\Lambda p_{\Lambda,max}(\mu)\ ,$$

with the *rate* of convergence:

$$0 \leq p_\Lambda(\mu) - p_{\Lambda,max}(\mu) \leq \mathcal{O}((\ln V)/V),$$

see [15]. The *rate* of convergence proved by the *Approximating Hamiltonian Method*(AHM) is

$$0 \le p_\Lambda(\mu) - p_{\Lambda,max}(\mu) \le \mathcal{O}(1/\sqrt{V}) \ ,$$

see [10], [26].

Remark 2. Although in [10] and in [15] the use of *coherent states* is essential, the method of the last paper efficiently exploits the Peierls-Bogoliubov and Berezin-Lieb inequalities instead of the AHM. To be more flexible, this method covers also the case of *infinitely* many k-modes, provided the card$\{k : k \in I_\Lambda \subset \Lambda^*\} < c\ V^{1-\gamma}$, $\gamma > 0$, and it gives also more accurate estimates. The Bogoliubov c-Number Approximation is *exact* on the thermodynamic level (AHM) [6], .

2.8 The c-Number Approximation for Ideal Bose-Gas

The c-number substitution in the grand-canonical Hamiltonian $T_\Lambda(\mu) := T_\Lambda - \mu N_\Lambda$ is

$$T_\Lambda(\mu) \to T_\Lambda(c,\mu) = \sum_{k \in \Lambda^* \setminus \{0\}} (\varepsilon_k - \mu) b_k^* b_k - V\mu|c|^2$$

Then one gets for the pressures (note that $\mu < 0$ and $\varepsilon_{k=0} = 0$):

$$p[T_\Lambda(\mu)] = \frac{1}{\beta V} \ln \mathrm{Tr}_{\mathfrak{F}} \exp[-\beta T_\Lambda(\mu)] = \frac{1}{\beta V} \sum_{k \in \Lambda^*} \ln(1 - e^{-\beta(\varepsilon_k - \mu)})^{-1}$$

$$p[T_\Lambda(c,\mu)] = \frac{1}{\beta V} \sum_{k \in \Lambda^* \setminus \{0\}} \ln(1 - e^{-\beta(\varepsilon_k - \mu)})^{-1} + \mu|c|^2$$

$$0 \le p[T_\Lambda(\mu)] - p[T_\Lambda(c,\mu)] = \frac{1}{\beta V} \ln(1 - e^{\beta\mu})^{-1} - \mu|c|^2 =: \Delta_\Lambda(c,\mu)$$

Variational Principle: $\{c : \inf_c \lim_\Lambda \Delta_\Lambda(c,\mu)\} = \{c_*(\mu)\} \Rightarrow$
$c_*(\mu < 0) = 0 \vee (\mu c_*(\mu)) \mid_{\mu=0} = 0$. Hence, the BEC density is *not* defined.

2.9 Gauge Invariance and Bogoliubov Quasi-Averages

Since $[H_\Lambda, N_\Lambda] = 0$ (*total particle number conservation law*),

$$H_\Lambda = e^{i\varphi N_\Lambda} H_\Lambda e^{-i\varphi N_\Lambda} \ , \ U(\varphi) := e^{i\varphi N_\Lambda} \ ,$$

H_Λ is invariant w.r.t. gauge transformations $U(\varphi)$.

Corollary 1. The grand-canonical expectation value:
$$\left\langle b_0/\sqrt{V}\right\rangle_{H_\Lambda}(\beta,\mu)=0.$$
Let $H_{\Lambda,\nu}(\mu):=H_\Lambda(\mu)-\sqrt{V}(\nu\ b_0^*+\nu^* b_0)$, $\nu\in\mathbb{C}$. Then
$$\left\langle \frac{b_0}{\sqrt{V}}\right\rangle_{H_{\Lambda,\nu}}(\beta,\mu)\neq 0\ ,\left\langle \frac{b_{k\neq 0}}{\sqrt{V}}\right\rangle_{H_{\Lambda,\nu}}(\beta,\mu)=0\ .$$

Remark 4. Whether the limit: $\lim_{\nu\to 0}\lim_\Lambda\left\langle \frac{b_0}{\sqrt{V}}\right\rangle_{H_{\Lambda,\nu}}(\beta,\mu)=:c_0\neq 0$?
If it is the case this yields a spontaneous breaking of the gauge symmetry. Here c_0 is the Bogoliubov *quasi-average* [4], [5]. The idea of quasi-averages allowed Bogoliubov to prove his famous $1/q^2$-Theorem for interacting Bose-gas as well as to advance later in elucidating the c-Number Approximation, see [15], [16], [21], [26].

Example 2.(Ideal Bose-Gas) The gauge-breaking sources imply
$$\begin{aligned}&T_{\Lambda,\nu}(\mu):=T_\Lambda(\mu)-\sqrt{V}(\nu\ b_0^*+\nu^* b_0)=\\&-\mu(b_0^*+\sqrt{V}\overline{\nu}/\mu)(b_0+\sqrt{V}\nu/\mu)+T_\Lambda^{(k\neq 0)}(\mu)+V|\nu|^2/\mu.\end{aligned}$$

The c-number substitution gives:
$$T_{\Lambda,\nu}(\mu)\to T_{\Lambda,\nu}(c,\mu)=-\mu V(\overline{c}+\overline{\nu}/\mu)(c+\nu/\mu)+T_\Lambda^{(k\neq 0)}(\mu)+V|\nu|^2/\mu$$

One gets for the pressure (note that $\mu<0$ and $\varepsilon_{k=0}=0$):
$$\begin{aligned}&p[T_{\Lambda,\nu}(\mu)]=p[T_\Lambda(\mu)]-|\nu|^2/\mu\ ,\\&p[T_{\Lambda,\nu}(c,\mu)]=p[T_\Lambda^{(k\neq 0)}(\mu)]+\mu V(\overline{c}+\overline{\nu}/\mu)(c+\nu/\mu)-|\nu|^2/\mu\ ,\\&0\leq p[T_{\Lambda,\nu}(\mu)]-p[T_{\Lambda,\nu}(c,\mu)]=\\&\frac{1}{\beta V}\ln(1-e^{\beta\mu})^{-1}-\mu|c+\eta/\mu|^2=:\Delta_{\Lambda,\nu}(c,\mu)\ .\end{aligned}$$

The Variational Principle:$\{c:\inf_c\lim_\Lambda\Delta_{\Lambda,\nu}(c,\mu)\}=\{c_*(\mu,\nu)=-\nu/\mu\}$ implies that the variational BEC density ρ_{0*} is defined by the limit $|\nu/\mu(\nu)|\underset{\nu\to 0}{\longrightarrow}\sqrt{\rho_{0*}}$ or equivalently by
$$\rho_{0*}:=\lim_{\substack{\nu\to 0\\ \mu=\mu(\nu)\to 0}}|c_*(\mu,\nu)|^2=\lim_{\substack{\nu\to 0\\ \mu=\mu(\nu)\to 0}}\lim_{V\to\infty}\langle\frac{b_0^*}{\sqrt{V}}\rangle_{T_{\Lambda,\nu}(\mu)}\langle\frac{b_0}{\sqrt{V}}\rangle_{T_{\Lambda,\nu}(\mu)}\ .$$

The relation of BEC versus the *quasi-average* BEC and the maximizer ρ_{0*} takes the form:

$$\text{zero}-\text{mode BEC } \rho_0 \Rightarrow \frac{1}{V}\langle b_0^* b_0\rangle_{T_{\Lambda,\nu=0}(\mu)} = \frac{1}{V}\frac{1}{e^{-\beta\mu}-1} \leq$$
$$\frac{|\nu|^2}{\mu^2} + \frac{1}{V}\frac{1}{e^{-\beta\mu}-1} = \frac{1}{V}\langle b_0^* b_0\rangle_{T_{\Lambda,\nu}(\mu)} \Rightarrow \text{quasi}-\text{average BEC} .$$

Then by the Variational Principle for the c-Number Approximation one obtains:

$$\lim_{\substack{\nu\to 0\\ \mu=\mu(\nu)\to 0}} \lim_{V\to\infty} \frac{1}{V}\langle b_0^* b_0\rangle_{T_{\Lambda,\nu}(\mu)} \overset{!}{=} \lim_{\substack{\nu\to 0\\ \mu=\mu(\nu)\to 0}} \lim_{V\to\infty} \langle \frac{b_0^*}{\sqrt{V}}\rangle_{T_{\Lambda,\nu}(\mu)} \langle \frac{b_0}{\sqrt{V}}\rangle_{T_{\Lambda,\nu}(\mu)}$$
$$\Rightarrow \text{gauge}-\text{symmetry breaking BEC} = \lim_{\substack{\nu\to 0\\ \mu=\mu(\nu)\to 0}} |c_*(\mu,\nu)|^2 = \rho_{0*} .$$

Remark 5. Is it possible that $\rho_0 < \rho_{0*}$? The answer is positive: one can prove this inequality for the *ideal* as well as for an *interacting* Bose-gas [7] if they manifest *generalised* BEC of the type II or III.

Proposition 4 [15], [16]. The $k = 0$ – mode BEC $\Rightarrow$ quasi-average BEC $\Leftrightarrow$ spontaneous gauge-symmetry breaking BEC $\Leftrightarrow$ non-zero c-number approximation for the mode $k = 0$.

The proof is based on Griffith's arguments and on the following two Propositions:

Proposition 5 For a real ν one gets equality between the limits:

$$\lim_{\Lambda} p_\Lambda(\mu;\nu) = \lim_{\Lambda} p'_\Lambda(\mu;\nu) = \lim_{\Lambda} p_{\Lambda,max}(\mu;\nu) ,$$

which are convex in ν.

Proposition 6 (Gauge-Symmetry Breaking and BEC)

$$\lim_{|\nu|\to 0,\ \arg(\nu)} \lim_{\Lambda} \left\langle \frac{b_0}{\sqrt{V}} \right\rangle_{H_{\Lambda,\nu}} (\beta,\mu) =$$
$$\lim_{|\nu|\to 0,\ \arg(\nu)} \lim_{\Lambda} |z_{\Lambda,max}(\nu)|\ e^{i\ \arg(\nu)}/\sqrt{V} =: c_0 .$$

Here by the Variational Principle: $z_{\Lambda,max}(\nu) = |z_{\Lambda,max}(\nu)|\ e^{i\ \arg(\nu)}$,

$$\sup_{\zeta} \mathrm{Tr}_{\mathfrak{F}'_0} \exp[-\beta H'_\Lambda(\zeta,\mu;\nu)] = \mathrm{Tr}_{\mathfrak{F}'_0} \exp[-\beta H'_\Lambda(z_{\Lambda,max}(\nu),\mu;\nu)]$$
$$= \exp\,[\beta V p_{\Lambda,z_{\Lambda,max}(\nu)}(\mu;\nu)] =: \exp\,[\beta V p_{\Lambda,max}(\mu;\nu)],$$

and $z_{\Lambda,max}(0) = |z_{\Lambda,max}(0)|\, e^{i\phi}$, $p_{\Lambda,z_{\Lambda,max}(\nu)}(\mu;\nu)|_{\nu=0} = p_{\Lambda,max}(\mu)$.

Corollary 2. One obtains for the quasi-average condensate density and for the condensate density equation:

$$\rho_0(\beta,\mu) = \lim_{|\nu|\to 0,\ \arg(\nu)} \lim_{\Lambda} \left\langle \frac{b_0^* \, b_0}{V} \right\rangle_{H_{\Lambda,\nu}} (\beta,\mu) = \lim_{\Lambda} |c_{0,\Lambda,max}|^2(\beta,\mu) \ .$$

where $c_{0,\Lambda,max}$ is a maximizer of the *variational problem*:

$$\sup_{c_0} \mathrm{Tr}_{\mathfrak{F}_0'} \exp[-\beta H_\Lambda'(c_0\sqrt{V},\mu)] = \mathrm{Tr}_{\mathfrak{F}_0'} \exp[-\beta H_\Lambda'(c_{0,\Lambda,max}\sqrt{V},\mu)]$$

3 Random Homogeneous (Ergodic) External Potentials.

3.1 Random and Kinetic-Energy Eigenfunctions

For the *almost surely* (a.s.) self-adjoint random Schrödinger operator in $\Lambda \subset \mathbb{R}^d$ one has:

$$h_\Lambda^\omega \, \phi_j^\omega = (t_\Lambda + v^\omega)_\Lambda \, \phi_j^\omega = E_j^\omega \, \phi_j^\omega \ , \text{ for almost all(a.a.) } \omega \in \Omega \ ,$$

where $\{\phi_j^\omega\}_{j\geq 1}$ are the *random* eigenfunctions. In the limit $\Lambda \uparrow \mathbb{R}^d$ the spectrum $\sigma(h^\omega)$ of this operator is a.s. nonrandom [19].

Let $N_\Lambda(\phi_j^\omega)$ be particle-number operator in the eigenstate ϕ_j^ω.

$$N_\Lambda := \sum_{j\geq 1} N_\Lambda(\phi_j^\omega) := \sum_{j\geq 1} b^*(\phi_j^\omega) b(\phi_j^\omega)$$

is the *total* number operator in the boson Fock space $\mathfrak{F}(\mathcal{L}^2(\Lambda))$, $b(\phi_j^\omega) := \int_\Lambda dx\, \overline{\phi_j^\omega}(x)\, b(x)$, and $\{\phi_j^\omega\}_{j\geq 1}$ is a.s. a (random) basis in $\mathcal{H} = \mathcal{L}^2(\Lambda)$.

Let $t_\Lambda \, \psi_k = \varepsilon_k \, \psi_k$ be the kinetic-energy operator eigenfunctions $\{\psi_k\}_{k\in\Lambda^*}$ with eigenvalues $\varepsilon_k = \hbar^2 \, k^2/2m$. Recall that one of the *key hypothesis* of the conventional Bogoliubov Theory is the existence of translation-invariant *ground-state* (i.e. the zero-mode $\psi_{k=0}$) Bose condensation.

Random Hamiltonian H_Λ^ω of interacting Bosons in $\mathfrak{F}(\mathcal{H})$:

$$H_\Lambda^\omega := T_\Lambda^\omega + U_\Lambda \ = \ \text{random Schrodinger operator} + \text{interaction} \ ,$$

where the kinetic-energy operator has two forms:

$$\mathrm{d}\Gamma(h^{\omega}_{\Lambda}) := T^{\omega}_{\Lambda} = \sum_{j\geq 1} E^{\omega}_{j}\ b^{*}(\phi^{\omega}_{j})b(\phi^{\omega}_{j}) = \sum_{k_1,k_2\in\Lambda^{*}} (\psi_{k_1},(t_{\Lambda}+v^{\omega})\psi_{k_2})_{\mathcal{H}} b^{*}_{k_1} b_{k_2} .$$

Note that there are also *two faces* for the second-quantised two-body interaction $u(x-y)$ in $\mathfrak{F}(\mathcal{H})$:

$$\begin{aligned} U_{\Lambda} &:= \frac{1}{2}\sum_{\substack{j_1,j_2\\ j_3,j_4}} (\phi^{\omega}_{j_1}\otimes\phi^{\omega}_{j_2}, u\ \phi^{\omega}_{j_3}\otimes\phi^{\omega}_{j_4})_{\mathcal{H}\otimes\mathcal{H}}\ b^{*}(\phi^{\omega}_{j_1})b^{*}(\phi^{\omega}_{j_2})\ b(\phi^{\omega}_{j_3})b(\phi^{\omega}_{j_4}) \\ &= \frac{1}{2V}\sum_{k_1,k_2,q\in\Lambda^{*}} v\,(q)\, b^{*}_{k_1+q} b^{*}_{k_2-q} b_{k_2} b_{k_1} \end{aligned}$$

Remark 6. Our aim is to elucidate the status and in particularly exactness of the Bogoliubov c-Number Approximation for the random interacting boson gas. For example to answer the questions concerning the (generalised) BEC:
$\sum_{j:E^{\omega}_{j}\leqslant\delta}\langle N_{\Lambda}(\phi^{\omega}_{j})\rangle_{H^{\omega}_{\Lambda}}/V \to c$ **?** or $\sum_{k:\varepsilon_k\leqslant\gamma}\langle N_{\Lambda}(\psi_k)\rangle_{H^{\omega}_{\Lambda}}/V \to c$ **?**

3.2 Random versus Kinetic-Energy Condensation

Proposition 7 [11] Let $H^{\omega}_{\Lambda} := T^{\omega}_{\Lambda} + U_{\Lambda}$ be *many-body* Hamiltonian of interacting bosons in random external potential V^{ω}_{Λ}. If the particle *interaction* U_{Λ} commutes with **any** of number operators $N_{\Lambda}(\phi^{\omega}_{j})$ (*local gauge invariance*), then

$$a.s.-\lim_{\delta\downarrow 0}\lim_{\Lambda}\sum_{j:E^{\omega}_{j}\leqslant\delta}\frac{1}{V}\langle(N_{\Lambda}(\phi^{\omega}_{j})\rangle_{H^{\omega}_{\Lambda}} > 0 \Leftrightarrow$$

$$\Leftrightarrow a.s.-\lim_{\gamma\downarrow 0}\liminf_{\Lambda}\sum_{k:\varepsilon_k\leqslant\gamma}\frac{1}{V}\langle N_{\Lambda}(\psi_k)\rangle_{H^{\omega}_{\Lambda}} > 0\ ,$$

and: $\lim_{\gamma\downarrow 0}\lim_{\Lambda}\sum_{k:\varepsilon_k>\gamma}\ \langle N_{\Lambda}(\psi_k)\rangle_{H^{\omega}_{\Lambda}}/V = 0$. Here $\langle - \rangle_{H^{\omega}_{\Lambda}}$ is quantum Gibbs expectation with random Hamiltonian H^{ω}_{Λ}.
Remark 7 If a many-body interaction satisfies the *local* gauge invariance:

$$[U_{\Lambda}, N_{\Lambda}(\phi_j)] = 0\ ,$$

then U_{Λ} is a *function* of the *occupation number operators* $\{N_{\Lambda}(\phi_j)\}_{j\geq 1}$. For this reason it is called a "*diagonal interaction*".

Corollary 3 A *random* localized generalised (of a yet unknown type) boson condensation occurs *if and only if* there is a generalised (type II/III) condensation in the extended (*kinetic-energy*) eigenstates. This is *a possible way* to save the Bogoliubov theory in a the case of *non-translation invariant*, but *homogeneous* random external potential.

3.3 Amounts of Random and of Kinetic-Energy Condensates

Let for any $A \subset \mathbb{R}_+$ the particle occupation measures m_Λ and $\tilde{m}_\Lambda$ are defined for the perfect Bose-gas by:

$$m_\Lambda(A) := \frac{1}{V} \sum_{j:E_i \in A} \langle N_\Lambda(\phi_i^\omega) \rangle_{T_\Lambda^\omega} \ , \ \tilde{m}_\Lambda(A) := \frac{1}{V} \sum_{k:\varepsilon_k \in A} \langle N_\Lambda(\psi_k) \rangle_{T_\Lambda^\omega} \ .$$

Proposition 8 [11] For the *perfect* Bose-gas amounts of random and kinetic-energy condensates coincide:

$$m(dE) \ = \left\{ \begin{array}{ll} (\overline{\rho} - \rho_c)\delta_0(dE) \ + \ (e^{\beta E} - 1)^{-1} \mathcal{N}(dE) & \text{if } \overline{\rho} \geqslant \rho_c \ , \\ (e^{\beta(E-\mu_\infty)} - 1)^{-1} \mathcal{N}(dE) & \text{if } \overline{\rho} < \rho_c \ , \end{array} \right.$$

$$\tilde{m}(d\varepsilon) \ = \left\{ \begin{array}{ll} (\overline{\rho} - \rho_c)\delta_0(d\varepsilon) \ + \ F(\varepsilon)d\varepsilon & \text{if } \overline{\rho} \geqslant \rho_c, \\ F(\varepsilon)d\varepsilon & \text{if } \overline{\rho} < \rho_c. \end{array} \right.$$

with *explicitly* defined density $F(\varepsilon)$. For models with *diagonal* interactions: $m_\Lambda(A) \leq \tilde{m}_\Lambda(A)$.

3.4 BEC in One-Dimensional Random Potential. Poisson Point-Impurities

For $d = 1$ and for repulsive *Poisson point-impurities* with density τ and $a > 0$, the homogeneous ergodic random external potential has the form:

$$\begin{aligned} v^\omega(x) : \ &= \int_{\mathbb{R}^1} \mu_\tau^\omega(dy) a \, \delta(x-y) = \sum_j a \, \delta(x - y_j^\omega) \\ \mathbb{P}\{\omega : \mu_\tau^\omega(\Lambda) = s\} \ &= \frac{|\Lambda|^s}{s!} e^{-\tau|\Lambda|} \ , \ \mathbb{E}(\mu_\tau^\omega(\Lambda)) = \tau|\Lambda| \ , \ \Lambda \subset \mathbb{R}^1 \ . \end{aligned}$$

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Proposition 9 [14] Let $a = +\infty$. Then $\sigma(h^\omega)$ is a.s. nonrandom, dense *pure-point* spectrum such that the closure $\overline{\sigma_{p.p.}(h^\omega)} = [0, +\infty)$, with the Integrated Density of States

$$\mathcal{N}(E) = \tau \frac{e^{-\pi\tau/\sqrt{2E}}}{1 - e^{-\pi\tau/\sqrt{2E}}} \sim \tau e^{-\pi\tau/\sqrt{2E}}\,,\ E \downarrow 0, \text{(Lifshitz tail)}.$$

One gets for the *spectrum*:

$$(a.s.) - \sigma(h^\omega) = \bigcup_j \left\{\pi^2 s^2 / 2(L_j^\omega)^2\right\}_{s=1}^{\infty}\ ,$$

where intervals $L_j^\omega = y_j^\omega - y_{j-1}^\omega$ are *independent identically distributed random variables* :

$$dP_{\tau, j_1, \ldots, j_k}(L_{j_1}, \ldots, L_{j_k}) = \tau^k \prod_{s=1}^{k} e^{-\tau L_{j_s}} dL_{j_s}$$

The *eigenfunctions:* for a.a. $\omega \in \Omega$ the one-particle *localized* quantum states $\{\phi_j^\omega\}_{j\geq 1}$, give a basis in $L^2(\Lambda)$.

4 Generalized c-numbers approximation

4.1 Existence of the approximating pressure

Since randomness implies *fragmented* (or generalized type II/III) condensation, following the Bogoliubov approximation philosophy, we want to replace all creation/annihilation operators in the momentum states ψ_k with kinetic energy *less* than some $\delta > 0$ by c-numbers. Let $I_\delta \subset \Lambda^*$ be the set of all *replaceable* modes

$$I_\delta := \left\{k \in \Lambda^* : \hbar^2 k^2 / 2m \leqslant \delta\right\},$$

and we denote $n_\delta := \text{card}\{k : k \in I_\delta\}$.

Remark 8 The number of quantum states n_δ is of the *order* V_l since by definition of the Integrated Density of States: $n_\delta = V\ \mathcal{N}_\Lambda(\delta)$. To use the *Lieb-Seiringer-Yngvason method* we consider $n_{\delta_\Lambda} = O(V^{1-\gamma})$, $0 < \gamma < 1$. Why it is possible ? See Corollary 4, and [25] for details.

4.2 Generalised BEC of type III: one-mode particle occupations

Definition 4 [12] We call eigenfunctions: $\{\phi_j^\omega\}_{j\geq 1}$ *weakly* localised if

$$\lim_{\Lambda} \frac{1}{\sqrt{V}} \int_{\Lambda} dx \; |\phi_j^\omega(x)| = 0 \;\; \text{for a.a. } \omega \in \Omega \; .$$

Proposition 10 [12],[13] Let all $\{\phi_j^\omega\}_{j\geq 1}$ be localised. Then for models H_Λ^ω with *diagonal interactions*

$$\lim_{\Lambda} \frac{1}{V} \langle N_\Lambda(\psi_k)\rangle_{H_\Lambda^\omega} = 0 \text{ for } \mathbf{all} \; k \in \{\Lambda^*\}$$

This implies that any possible generalised *kinetic-energy* BEC in these models is of *type III*.
Corollary 4 The number of *condensed* kinetic-modes is at most $O(V^{1-\gamma})$, $0 < \gamma < 1$, and in this case one can use the LSY method for the modes:

$$\lim_{\Lambda} \frac{1}{V^\gamma} \langle N_\Lambda(\psi_k)\rangle_{H_\Lambda^\omega} \neq 0 \; , \;\; \text{for k} \in \mathrm{I}_{\delta_\Lambda}, \;\; \gamma = 1 - \epsilon$$

Let $\mathcal{H}^\delta$ be the subspace of $\mathcal{H}$ spanned by the set of ψ_k with $k \in I_\delta$, and P_δ be orthogonal projector onto this subspace. Hence, we have a natural decomposition of the total space $\mathcal{H}$ and the corresponding representation for the associated symmetrised Fock space:

$$\mathcal{H} = \mathcal{H}^\delta \oplus \mathcal{H}' \; , \qquad \mathfrak{F} \approx \mathfrak{F}^\delta \otimes \mathfrak{F}'.$$

Then we proceed with the Bogoliubov substitution $b_k \to c_k$ and $b_k^* \to \bar{c}_k$ for all $k \in I_\delta$, which provides an *approximating* (for the initial) Hamiltonian, that we denote by $H_\Lambda^{low}(\mu, \{c_k\})$.

The partition function and the corresponding pressure for this *approximating* Hamiltonian have the form:

$$\Xi_\Lambda^{low}(\mu, \{c_k\}) = \mathrm{Tr}_{\mathfrak{F}'} \, e^{-\beta H_\Lambda^{low}(\mu, \{c_k\})} \; ,$$

$$p_{\Lambda,\delta}^{low}(\mu, \{c_k\}) = \frac{1}{V} \ln \Xi_\Lambda^{low}(\mu, \{c_k\}) \; .$$

Proposition 10 [13], [25] The c-numbers substitution for all operators in the energy-band I_{δ_Λ}, $\mathrm{card}\{k : k \in I_{\delta_\Lambda}\} = O(V^{1-\gamma})$, does not affect the original pressure in the following sense:

$$\text{a.s.}-\lim_{\Lambda}[p_\Lambda(\beta,\mu) - \{\max_{\{c_k\}} p_{\Lambda,\delta_\Lambda}^{low}(\mu, \{c_k\})\}] = 0$$

Remark 9 Besides the *type III* condensation the last statement covers the one-mode case. For the case of eventual *type II* condensation the arguments are similar, but with a volume-dependent cut-off of the converging sum over modes [25].

References

[1] M.Beau, V.A.Zagrebnov. *The second critical density and anisotropic generalised condensation.* Cond. Matter Phys., **31**, 23003:1-10, (2010).

[2] N.N.Bogoliubov. *About the theory of superfluidity.* Izv. Akad. Nauk USSR. **11**, (1947), 77–90.

[3] N.N.Bogoliubov. *Energy levels of the imperfect Bose-Einstein gas.* Bull. Moscow State Univ., **7**, (1947), 43–56.

[4] N.N.Bogoliubov. *Quasiaverages in the Problems of Statistical Mechanics.* Communications of JINR-Dubna **D-781**, (1961), 1–123, and in: Selected Papers, vol.3, Kiev: Naukova Dumka, 1971, 174–243 see also in: Collection of Sinetific Works (in twelve volumes): Statistical Mechanics, vol.6, Mocow: Nauka, 2006, 236–360.

[5] N.N.Bogoliubov. *Lectures on Quantum Statistics* (Vol.2): Quasi-Averages. New York: Gordon and Breach, 1970.

[6] N.N.Bogolyubov (jr), J.Brankov, V.A.Zagrebnov, A.M.Kurbatov, and N.Tonchev. *Some classes of exactly soluble models of problems in Quantum Statistical Mechanics: The method of approximating Hamiltonian.* Russian Math. Surveys **39**, (1984), 1–50 .

[7] E.Buffet, Ph. de Smedt, J.V.Pulè, *The condensate equation for some Bose systems.* J. Phys. A: Math.Gen., **16**, (1983), 4307–4324.

[8] H.B.G.Casimir. *On Bose-Einstein condensation.* In: Fundamental Problems in Statistical Mechanics III, ed. E.G.D. Cohen. (Amsterdam: North-Holland Publishing Company, 1968), 188–196.

[9] A.Einstein. *Quantentheorie des einatomigen idealen Gases.* Sitzungsberichte der Preussischen Akademie der Wissenschaften, **I**, (1925), 3–14.

[10] J. Ginibre. *On the Asymptotic Exactness of the Bogoliubov Approximation for Many Boson Systems.* Commun.Math.Phys., **8**, (1968), 26–51.

[11] T.Jaeck, J.V.Pulé, V.A.Zagrebnov. *On the nature of Bose-Einstein condensation in disordered systems.* J. Stat. Phys., **137**, (2009), 19–55.

[12] T.Jaeck, J.V.Pulé, V.A.Zagrebnov. *On the nature of Bose-Einstein condensation enhanced by localization.* J.Math.Phys., **51**, (2010), 103302-1-15.

[13] T.Jaeck, V.A.Zagrebnov. *Exactness of the Bogoliubov approximation in randomexternal potentials.* J.Math.Phys., **51**, (2010), 123306-1-16.

[14] O.Lenoble, V.A.Zagrebnov. *Bose-Einstein condensation in the Luttinger-Sy model.* Markov Process. Rel. Fields, **13**, (2007), 441–468.

[15] E.H.Lieb, R.Seiringer, J.Yngvason. *Justification of c-Number Substitutions in Bosonic Hamiltonians.* Phys.Rev.Lett., **94**, (2005), 0804401–4.

[16] E.H.Lieb, R.Seiringer, J.Yngvason. *Bose-Einstein Condensation and Spontanious Symmetry Breaking.* Rept.Math.Phys., **59**, (2007), 389–399.

[17] F.London. *On the Bose-Einstein condensation.* Phys. Rev., **54**, (1938), 947–954.

[18] W.J.Mullin, A.R.Sakhel. *Generalized Bose-Einstein Condensation.* J.Low Temp.Phys., **166**, (2012), 125–150.

[19] L.A.Pastur, A.Figotin, *Spectra of Random and Almost-Periodic Operators.* Berlin: Springer-Verlag, 1992.

[20] D.Ya.Petrina, *Mathematical Foundations of Quantum Statistical Mechanics (Continuous Systems).* Dordrecht: Kluwer, 1995.

[21] A.Sütő. *Equivalence of Bose-Einstein Condensation and Symmetry Breaking.* Phys.Rev.Lett., **94**, (2005), 0804402–4.

[22] M. Van den Berg. *On boson condensation into an infinite number of low-lying levels.* J. Math. Phys., **23**, (1982), 1159–1161.

[23] M.Van den Berg, J.T.Lewis. *On generalized condensation in the free boson gas.* Physica, **110A**, (1982), 550–564.

[24] M.Van den Berg, J.T.Lewis, J.V.Pulé. *A general theory of Bose-Einstein condensation.* Helv.Phys.Acta, **59**, (1986), 1271–1288.

[25] V.A.Zagrebnov. *Lectures on the Bose-Einstein Condensation Theory.* Università degli Studia Roma Tre, 2013.

[26] V.A.Zagrebnov, J.-B.Bru. *The Bogoliubov model of Weakly Imperfect Bose Gas.* Phys.Rept., **350**, (2001), 291–434.